We present a generalized version of the Feynman-Vernon-Hellwarth geometric representation and a biorthogonal density matrix formalism for the description of the non-Hermitian Schrödinger equation. The theory is applied to the study of complex geometric quantum phases in dissipative systems. It is shown that the complex Aharonov-Anandan (AA) geometric phase is related to the complex solid angle enclosed by a complex Bloch vector trajectory $S(t)$. General analytic formulas are presented for the complex AA phase for a driven dissipative two-level system undergoing multiphoton Rabi floppings.

1. Introduction

The study of the geometric phase factor accompanied by an adiabatic or cyclic change in quantum and classical systems has received considerable attention recently [1-6]. The adiabatic quantum phase discovered by Berry [1] is associated with the adiabatic evolution of a Hamiltonian $\hat{H}(R)$ along a closed curve $I$ in the parameter ($R$) space. When a quantum state remains an eigenstate of $\hat{H}(R)$ corresponding to a simple eigenvalue $E(R)$ during the cyclic evolution, a geometric phase (known as the Berry phase) develops which depends only on $I$. More recently, Aharonov and Anandan [3] have introduced a new cyclic quantum phase, called the AA phase, which is a gauge-invariant generalization of the Berry phase without recourse to adiabaticity. The AA phase is a more general concept and is associated with the evolution of any cyclic state, i.e. a quantum state $|\psi(t)\rangle$ which returns to itself, apart from a phase factor, after some time $T$: $|\psi(T)\rangle = \exp(i\Phi) |\psi(0)\rangle$. Both Berry and AA phases have recently been experimentally detected by means of optical phenomena [7].

With the exception of one work [6], however, all the studies of geometric phase factors so far have been confined to the evolution of unitary operators or Hermitian Hamiltonians. In this paper, we present a density matrix formulation of complex geometric phases for dissipative quantum systems involving non-Hermitian Hamiltonians. Since both $|\psi(t)\rangle$ and the conventional density matrix defined by $\rho(t) = |\psi(t)\rangle \langle \psi(t)|$ are decaying in time, a cyclic state cannot be defined. In section 2, we present a generalized density matrix formulation which avoids this difficulty. In addition, we show how the celebrated Feynman-Vernon-Hellwarth geometric representation [8] of the Hermitian Schrödinger equation may be extended to the case of the non-Hermitian Schrödinger equation. This provides a natural framework for the description of complex geometric phases in dissipative systems. In section 3, we prove a theorem which relates the complex geometric phase to a complex solid angle. Finally we apply the formalism to multiphoton Rabi floppings in two-level systems and present general analytic formulas for complex (as well as real) geometric phases in section 4.

2. Geometric representation of non-Hermitian Schrödinger equation

Consider the following time-dependent Schrödinger equation ($\hbar = 1$):

$$0 0 0 9-2 6 1 4 / 8 9 / 0 3 . 5 0 \odot E l s e v i e r S c i e n c e P u b l i s h e r s B . V .$$

( North-Holland Physics Publishing Division )
\[ i\langle d/dt | \psi(t) \rangle = \hat{H}(t) \langle \psi(t) \rangle, \]  
(1)

where

\[ \hat{H}(t) = \hat{H}_0 + \hat{V}(t). \]  
(2)

and

\[ \hat{H}(t) = \hat{H}_0 + \hat{V}(t). \]  
(3)

\( \hat{H}_0 \) is the unperturbed Hamiltonian of the two-level system with eigenstates \( |\alpha\rangle \) and \( |\beta\rangle \) and eigenvalues \( E_\alpha \) and \( E_\beta \), and \( \hat{V}(t) \) is the perturbation. In eq. (2), \( \hat{G} \) is the diagonal damping operator with eigenvalues \( g_\alpha \) and \( g_\beta \) [9,10]:

\[ \hat{G}|\gamma\rangle = g_\gamma|\gamma\rangle \quad (\gamma = \alpha \text{ or } \beta). \]  
(4)

g_\gamma can represent, for example, the spontaneous decay rate of level \( |\gamma\rangle \), etc.

To construct the density matrix, the conventional way [4,10] is to adopt

\[ \hat{\rho}'(t) = |\psi(t)\rangle \langle \psi(t)|, \]  
(5)

where \( |\psi(t)\rangle \) is the solution of eq. (1). This leads to the Liouville equation of the following form [10]:

\[ i\langle d/dt | \hat{\rho}'(t) \rangle = [\hat{H}(t), \hat{\rho}'(t)] - i[\hat{G}, \hat{\rho}'(t)]_+, \]  
(6)

where

\[ [\hat{A}, \hat{B}]_+ = \hat{A}\hat{B} - \hat{B}\hat{A} \quad \text{and} \quad \{\hat{A}, \hat{B}\}_+ = \hat{A}\hat{B} + \hat{B}\hat{A}. \]

Due to the dissipative \( \{ , \}_+ \) term, the density matrix \( \hat{\rho}' \) described by eq. (6) does not have a conserved norm and its trace, \( \text{Tr} \hat{\rho}'(t) \), is decreasing in time. This causes difficulty in the description of the geometric phase as the density matrix is required to return to its initial value after a cyclic evolution of the system [3].

To overcome the difficulty, we consider the following generalized density matrix:

\[ \hat{\rho}(t) = |\psi(t)\rangle \langle \chi(t)|, \]  
(7)

defined by the biorthonormal Hilbert space [11]. Here \( |\chi(t)\rangle \) is the solution of the Schrödinger equation with the adjoint Hamiltonian \( \hat{H}^+(t) \)

\[ i\langle d/dt | \chi(t) \rangle = \hat{H}^+(t) |\chi(t)\rangle. \]  
(8)

Eq. (7) leads to the following generalized Liouville equation:

\[ i\langle d/dt | \hat{\rho}(t) \rangle = [\hat{H}(t), \hat{\rho}(t)] \]  
(9)

the form of which is identical to the ordinary Liouville equation without dissipation! Further, in the biorthonormal Hilbert space, one has

\[ \text{Tr} \hat{\rho}(t) = \langle \chi(t) | \psi(t) \rangle = \langle \chi(0) | \psi(0) \rangle = 1, \]  
(10)

and the norm of the three-vector (to be defined below) is conserved in time even as the system is dissipating.

The solutions of eqs. (1) and (8) can be written, for a two-level system, in the general form

\[ |\psi(t)\rangle = a(t) |\alpha\rangle + b(t) |\beta\rangle, \quad \langle \chi(t) | = a(t) \langle \alpha| + \delta(t) \langle \beta|. \]  
(11)

In terms of column and row vectors, we have

\[ |\psi(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \quad \langle \chi(t) | = (a(t) \delta(t)). \]  
(12)
subject to the biorthogonal relation,
\[ \vec{a}(t) \cdot \vec{b}(t) = \delta(t) \]
(13)

Following Feynman et al. [8], we now construct the three-vector \( \vec{r} \) as
\[
\begin{align*}
\vec{r}_1(t) &= u(t) = \text{Tr}(\hat{\rho}(t) \hat{\sigma}_x) = a \delta + b \alpha, \\
\vec{r}_2(t) &= v(t) = \text{Tr}(\hat{\rho}(t) \hat{\sigma}_y) = i(a \delta - b \alpha), \\
\vec{r}_3(t) &= w(t) = \text{Tr}(\hat{\rho}(t) \hat{\sigma}_z) = a \alpha - b \beta,
\end{align*}
\]
(14)
where the \( \hat{\sigma} \)'s are the Pauli spin matrices, and \( \vec{r}(t) \) are complex quantities. (For non-dissipative systems, \( \dot{\vec{r}} = 0 \), \( \vec{r} \rightarrow \vec{r} * \), and \( \vec{r}(t) \) become real quantities [8].) From eqs. (14), one can readily show that the norm of the complex Bloch vector \( \vec{S}(t) = (u, v, w) \) is conserved in time,
\[
\vec{r}_1^2(t) + \vec{r}_2^2(t) + \vec{r}_3^2(t) = u^2(t) + v^2(t) + w^2(t) = 1.
\]
(15)
Thus \( \vec{S}(t) = (u, v, w) \) is a complex three-vector with unit norm and traces out a trajectory in the complex three-space. Further it can be shown that the differential equation for \( \vec{r} \) is
\[
\frac{d\vec{r}}{dt} = \Omega \times \vec{r},
\]
(16)
where \( \Omega \) is also a three-vector in "|r|" space defined by \( \Omega_1 = \text{Tr}(\hat{H}(t) \hat{\sigma}_x) \), \( \Omega_2 = \text{Tr}(\hat{H}(t) \hat{\sigma}_y) \), \( \Omega_3 = \text{Tr}(\hat{H}(t) \hat{\sigma}_z) \). Eqs. (7)-(16) are the generalization of the Feynman–Vernon–Hellwarth geometric representation to the non-Hermitian Schrödinger equation.

3. Complex geometric phase in dissipative two-level systems

Under a cyclic quantum evolution,
\[
\begin{align*}
\hat{\rho}(t+T) &= \hat{\rho}(t), \\
\hat{S}(t+T) &= \hat{S}(t)
\end{align*}
\]
(17)
and
\[
\begin{align*}
|\psi(t+T)\rangle &= \exp(i\Phi) |\psi(t)\rangle, \\
|\chi(t+T)\rangle &= \exp(i\Phi^*) |\chi(t)\rangle,
\end{align*}
\]
(18)
where \( \Phi (\Phi^*) \) is the total (complex) phase associated with the cyclic evolution of \( |\psi(t)\rangle \) (|\chi(t)\rangle). To evaluate \( \Phi \), we follow the procedure of Aharonov and Anandan [3,6] by introducing a modified adjoint pair
\[
\begin{align*}
|\widetilde{\psi}(t)\rangle &= \exp[-if(t)] |\psi(t)\rangle, \\
|\widetilde{\chi}(t)\rangle &= \exp[-if^*(t)] |\chi(t)\rangle.
\end{align*}
\]
(19)
Here \( f(t) \) is an arbitrary function satisfying \( f(t+T) - f(t) = \Phi \), and
\[
\begin{align*}
|\widetilde{\psi}(t+T)\rangle &= |\widetilde{\psi}(t)\rangle, \\
|\widetilde{\chi}(t+T)\rangle &= |\widetilde{\chi}(t)\rangle,
\end{align*}
\]
(20)
are periodic in time with period \( T \). Using eqs. (1) and (8), we get
\[
df/dt = \langle \widetilde{\chi} | i \hat{\delta} / \hat{\partial t} | \widetilde{\psi} \rangle - \langle \widetilde{\chi} | \hat{H} | \widetilde{\psi} \rangle
\]
(21)
and thus
\[
\Phi = \alpha_D + \beta_G
\]
(22)
with
\[
\alpha_D = \text{dynamical phase} = - \int_0^T \langle \chi | \hat{H} | \psi \rangle \, dt
\]
(23)
and

\[ \beta_G = \text{AA geometric phase} = \int_0^T \langle \tilde{\chi} | i \partial / \partial t | \tilde{\phi} \rangle \, dt. \]  

(24)

We shall now introduce the following Theorem.

Theorem. The complex geometric phase \( \beta_G \) defined in eq. (24) is equal to one-half of the complex solid angle \( \Omega(C) \) enclosed by the complex trajectory \( S(t) = (u, v, w) \).

Proof. The solid angle \( \Omega(C) \) enclosed by a closed curve \( C \) can be defined as

\[ \Omega(C) = \int_0^T \left[ 1 - \cos \theta(t) \right] \phi \, dt. \]  

(25)

We shall show that \( \beta_G = \frac{1}{2} \Omega(C) \). Assume \( (u, v, w) \) form an orthogonal three-vector, we can define

\[ \cos \theta = w = a\bar{a} - b\bar{b}, \quad \tan \phi = \frac{u}{v} = \frac{\bar{a}b + a\bar{b}}{i(ab - ba)}, \]  

(26)

where \( \theta \) and \( \phi \) are complex spherical angles. After some algebra and using the relations eq. (13) and

\[ 4a\bar{a}b\bar{b} = 1 - w^2, \]  

(27)

we find

\[ \phi = 2i \frac{b\bar{b}(a\bar{a} - \bar{a}a) + a\bar{a}(b\bar{b} - \bar{b}b)}{1 - w^2} \]  

(28)

and

\[ 1 - \cos \theta = 2\bar{b}b = 2(1 - a\bar{a}) \]  

(29)

Thus

\[ (1 - \cos \theta)\phi = i \frac{4(1 - a\bar{a})b\bar{b}(a\bar{a} - \bar{a}a) + 4b\bar{b}a\bar{a}(b\bar{b} - \bar{b}b)}{1 - w^2}. \]  

Using eq. (28) and the relation

\[ a\bar{a} + a\bar{a} + b\bar{b} + b\bar{b} = 0, \]  

(30)

we get

\[ (1 - \cos \theta)\phi = 2i \left[ (\hat{a}a + b\bar{b}) + \frac{1}{2} \left( \frac{\hat{a}}{a} - \frac{\hat{b}}{b} \right) \right]. \]  

(31)

Next we work out the geometric phase \( \beta_G \) from eq. (24). Since

\[ |\tilde{\phi} \rangle = \begin{pmatrix} a \exp(-if) \\ b \exp(if) \end{pmatrix} \quad \text{and} \quad \langle \tilde{\chi} | = \begin{pmatrix} \bar{a} \exp(if) & \bar{b} \exp(if) \end{pmatrix}, \]

we have

\[ \beta_G = i \int_0^T \langle \tilde{\chi} | d/dt | \tilde{\phi} \rangle \, dt = i \int_0^T dt \left( \hat{a}a + b\bar{b} \right) + \int_0^T f \, dt. \]  

(32)
The first term on the right-hand side is equal to \(-\alpha_D\) (dynamical phase), as can be shown from eq. (23). The second term is the total phase \(\Phi\), since
\[
\int_0^T \dot{f} dt = f(T) - f(0) = \Phi.
\]

Now from eqs. (18), we have, respectively,
\[
\Phi = i^{-1} \ln \left[ a(T)/a(0) \right] \quad \text{and} \quad \Phi = -i^{-1} \ln \left[ a(T)/a(0) \right].
\]

Hence
\[
\Phi = \frac{1}{i} \int_0^T [\dot{a}(t)/a(t) - \dot{a}(0)/a(0)] dt.
\]

Using eqs. (25), (31), (32), and (33), we finally arrive at
\[
\Omega(C) = 2\beta_G = 2(\Phi - \alpha_D). \quad \text{Q.E.D.}
\]

For non-dissipative systems, \(\theta\), \(\phi\), and therefore \(\beta_G\) and \(\Omega(C)\) become real quantities.

4. Geometric phases for multiphoton transitions

In this section, we shall present an example of a two-level dissipative system undergoing multiphoton Rabi floppings. Only the main results will be outlined here. A detailed treatment of geometric phases for multiphoton transitions will be presented elsewhere [12].

Consider the time evolution of the Schrödinger equation, eq. (1), for a two-level system driven by an intense periodic field. The perturbation \(\hat{V}(t)\) in eq. (3) is now given by the electric dipole interaction
\[
\hat{V}(t) = -\mu \epsilon_0 \cos(\omega t + \phi),
\]
where \(\mu\) is the electric dipole moment of the system and \(\epsilon_0\), \(\omega\), and \(\phi\) are, respectively, the field amplitude, frequency, and phase. In terms of the unperturbed bases \(\{|\alpha\rangle, |\beta\rangle\}\) of the two-level system, the total Hamiltonian \(\hat{H}(t)\) has the following matrix form
\[
\hat{H}(t) = \begin{bmatrix}
E_\alpha - ig_\alpha & V_{\alpha\beta}(t) \\
V_{\beta\alpha}(t) & E_\beta - ig_\beta
\end{bmatrix}
\]
where \(V_{\alpha\beta}(t) = \langle \alpha | \hat{V}(t) | \beta \rangle\), and \(g_\alpha\) \((g_\beta)\) is the damping constant. The non-Hermitian time-dependent Schrödinger equation, eq. (1), with periodic Hamiltonian \(\hat{H}(t+2\pi/\omega) = \hat{H}(t)\), given in eq. (36), can be transformed into an equivalent infinite-dimensional non-Hermitian Floquet matrix \((\hat{A}_F)\) eigenvalue problem [10,11,13],
\[
\hat{A}_F |\lambda_{yn}\rangle = \lambda_{yn} |\lambda_{yn}\rangle,
\]
where \(\lambda_{yn}\) and \( |\lambda_{yn}\rangle\) are the complex quasi-energy eigenvalues and eigenvectors, respectively, with \(\gamma = \alpha\) or \(\beta\) and \(n\) (Fourier index) = \(-\infty\) to \(+\infty\).

For nearly resonant multiphoton processes, \(E_\beta - E_\alpha = \omega_0 \approx (2n+1)\omega,\) \(n = 1, 2, \ldots\), the infinite-dimensional Floquet matrix \(\hat{A}_F\) can be further reduce to a two-by-two effective Hamiltonian by using appropriate nearly degenerate high-order perturbation theory [10,14].
\[ A_{\text{eff}} = \begin{bmatrix} E_\alpha + \delta_\alpha^{(c)} & u_{\alpha\beta} \\ \ast_{\beta\alpha} & E_\beta - (2n + 1)\omega + \delta_\beta^{(c)} \end{bmatrix} \]  

where \( \delta_\alpha^{(c)} \) (\( \delta_\beta^{(c)} \)) and \( u_{\alpha\beta} \) (\( u_{\beta\alpha} \)) are, respectively, the (complex) ac Stark shifts and effective couplings. For (2n+1)-photon transition, the leading terms in \( \delta \) and \( u \) can be derived using the (2n+1)-order nearly degenerate perturbation method [12]. This gives

\[ \delta_\alpha^{(c)} = -\delta_\beta^{(c)} = -|b|^2 \left[ \frac{2n+1}{2n(n+1)} - \frac{1}{4\omega^2} \frac{g_\alpha - g_\beta}{n(n+1)^2} \right] + O(b^4), \]

\[ u_{\alpha\beta} = B [1 - i(g_\alpha - g_\beta)S_n/2\omega] + O(b^{2n+3}) \]

and

\[ u_{\beta\alpha} = B^* [1 - i(g_\alpha - g_\beta)S_n/2\omega] + O(b^{2n+3}), \]

(39a)  

(39b)  

(39c)

with

\[ b = -\frac{1}{2} \langle \alpha | \mu | \beta \rangle \cdot \epsilon_0, \quad B = \frac{(-1)^n |b|^{2n} \exp[i(2n+1)\phi]}{2^{2n}(n!)^2\omega^{2n}}, \quad S_n = \sum_{k=1}^{n} 1/k. \]

For the one-photon (\( n=0 \)) case, we have \( \delta_\alpha^{(c)} = -\delta_\beta^{(c)} = -|b|^2/2\omega \), and \( S_0 = 0 \). The effective Hamiltonian \( \tilde{A}_{\text{eff}} \), eq. (38), possesses two complex eigenvalues \( \lambda_\pm \) and eigenvectors \( |\lambda_\pm \rangle \),

\[ \tilde{A}_{\text{eff}} |\lambda_\pm \rangle = \lambda_\pm |\lambda_\pm \rangle, \]

where

\[ \lambda_\pm = \kappa \pm q, \]

\[ \kappa = \frac{1}{2} \text{Tr}(\tilde{A}_{\text{eff}}), \quad q = \frac{1}{2} \sqrt{\Delta^2 + 4u_{\alpha\beta}u_{\beta\alpha}}, \]

(40)  

(41)  

(42)  

and \( \Delta \) is the detuning parameter,

\[ \Delta = E_\alpha - [E_\beta - (2n+1)\omega] + \delta_\alpha^{(c)} - \delta_\beta^{(c)}. \]

(43)

The quasienergy eigenvectors \( |\lambda_\pm \rangle \) are

\[ |\lambda_+ \rangle = \left[ \begin{array}{c} (u_{\alpha\beta}/u_{\beta\alpha})^{1/4} \cos(\frac{1}{4}\theta) \\ (u_{\beta\alpha}/u_{\alpha\beta})^{1/4} \sin(\frac{1}{4}\theta) \end{array} \right], \quad |\lambda_- \rangle = \left[ \begin{array}{c} - (u_{\alpha\beta}/u_{\beta\alpha})^{1/4} \sin(\frac{1}{4}\theta) \\ (u_{\beta\alpha}/u_{\alpha\beta})^{1/4} \cos(\frac{1}{4}\theta) \end{array} \right], \]

where \( \theta \) is a complex angle defined by

\[ \tan \theta = \frac{2\sqrt{u_{\alpha\beta}u_{\beta\alpha}}}{\Delta}. \]

(44)

(45)

Similarly, the complex eigenvalues \( \epsilon_\pm \) and eigenvectors \( |\epsilon_\pm \rangle \) of the adjoint Hamiltonian

\[ \tilde{A}_{\text{eff}}^+ |\epsilon_\pm \rangle = \epsilon_\pm |\epsilon_\pm \rangle, \]

can be obtained:

\[ \epsilon_\pm = \lambda_\pm^*, \]

(46)

(47)

and

\[ \langle \epsilon_+ | = \left[ (u_{\beta\alpha}/u_{\alpha\beta})^{1/4} \cos(\frac{1}{4}\theta) \right] \right. \left. (u_{\alpha\beta}/u_{\beta\alpha})^{1/4} \sin(\frac{1}{4}\theta) \right], \]

\[ \langle \epsilon_- | = \left[ -(u_{\beta\alpha}/u_{\alpha\beta})^{1/4} \sin(\frac{1}{4}\theta) \right] \right. \left. (u_{\alpha\beta}/u_{\beta\alpha})^{1/4} \cos(\frac{1}{4}\theta) \right]. \]

(48)
We note \( |\lambda_+\rangle \) and \( \langle \epsilon_+ | \) satisfy the following biorthonormal and closure relationships:

\[
\langle \epsilon_+ | \lambda_+ \rangle = 1, \quad \langle \epsilon_+ | \lambda_- \rangle = 0 \quad \text{and} \quad |\lambda_+\rangle \langle \epsilon_+ | + |\lambda_-\rangle \langle \epsilon_- | = I.
\]

The wavefunctions \( |\psi(t)\rangle \) and \( \langle \chi(t) | \) can now be approximated as

\[
|\psi(t)\rangle \approx \exp[-i\hat{A}_m(t-t_0)] |\psi(t_0)\rangle
= \exp[-i\kappa(t-t_0)] \{ \exp[-iq(t-t_0)] |\lambda_+\rangle \langle \epsilon_+ | + \exp[iq(t-t_0)] |\lambda_-\rangle \langle \epsilon_- | \psi(t_0)\rangle \},
\]

and

\[
\langle \chi(t) | \approx \langle \chi(t_0) | \exp[i\hat{A}_m(t-t_0)]
= \exp[i\kappa(t-t_0)] \{ \exp[iq(t-t_0)] \langle \chi(t_0) |\lambda_+\rangle \langle \epsilon_+ | + \exp[-iq(t-t_0)] \langle \chi(t_0) |\lambda_-\rangle \langle \epsilon_- | \}. \tag{49a}
\]

The complex Bloch vector \( \mathbf{S}(t) = (u, v, w) \) can be constructed according to eq. (14).

Following the procedure described in section 3, we arrive at the following general formula for the complex AA geometric phase for multiphoton Rabi floppings with period \( T = \pi q \),

\[
\beta_G^{(C)} = \pi \left[ 1 + \langle \chi(0) |\lambda_+\rangle \langle \epsilon_+ | \psi(0)\rangle - \langle \chi(0) |\lambda_-\rangle \langle \epsilon_- | \psi(0)\rangle \right],
\]

when the density matrix \( \hat{\rho}(t) = |\psi(t)\rangle \langle \chi(t) | \) returns to its initial value, \( \hat{\rho}(T) = \hat{\rho}(0) \). The corresponding formula for non-dissipative systems has the following simple form,

\[
\beta_G = \pi \left[ 1 + |\langle \lambda_+ | \psi(0)\rangle |^2 - |\langle \lambda_- | \psi(0)\rangle |^2 \right],
\]

where \( \beta_G \) is real.

As a simple application, consider a two-level system initially prepared in the ground state, \( |\psi(0)\rangle = |\alpha\rangle \), the laser phase is \( \phi = 0 \), and \( b \) is real. In this case \( u_{\alpha\beta} = u_{\beta\alpha} \), and eqs. (46), (47), and (50) give rise to the simple formula

\[
\beta_G^{(C)} = \pi \left( 1 + \cos \theta \right), \tag{52}
\]

where

\[
\cos \theta = \frac{\Delta}{\sqrt{\Delta^2 + 4u_{\alpha\beta}^2}},
\]

where \( \Delta \) and \( \theta \) are given in eqs. (39) and (43), respectively. In the special case of one-photon transition, \( n=0 \), and in the limit of rotating wave approximation (RWA), we have \( \Delta = (\omega - \omega_0) - i(g_\alpha - g_\beta) \), and \( u_{\alpha\beta} = b \). Eq. (52) becomes

\[
\beta_G^{(C)} = \pi \left( 1 + \frac{(\omega - \omega_0) - i(g_\alpha - g_\beta)}{\sqrt{[(\omega - \omega_0) - i(g_\alpha - g_\beta)]^2 + 4b^2}} \right), \tag{54}
\]

which is the AA phase analog of the Berry phase obtained in ref. [6].

We note that while our theoretical analysis presented in this paper is for the general Aharonov–Anandan complex geometric phase, similar formalism can be easily applied to the study of complex Berry phase as well. We are currently also extending a super-operator density matrix formalism [13] to the study of geometric phases in dissipative systems involving both \( T_1 \) (population damping) and \( T_2 \) (coherent damping) mechanisms. Work in these directions will be reported elsewhere [12].
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References