The quasi-energy eigenfunctions of N-level quantum systems driven by intense M-mode polychromatic fields are studied by means of the many-mode Floquet theory. It is found that the eigenfunctions exhibit striking self-similar (fractal) behavior in the temporal Fourier space. A method is presented for the calculation of the fractal ($D_f$) and entropy ($D_s$) dimensions of quasi-energy states and applied to the study of the behavior of two-level systems perturbed by bichromatic fields. The fractal dimensions are found to be (laser) intensity dependent and obey the relationship $D_f \ll D_s < M$.

Fractal geometry [1] provides both a description and a mathematical model for many of the seemingly complex forms and patterns in nature and the sciences. Shapes such as coastlines, mountains, and clouds are not easily described by traditional Euclidean geometry. Nevertheless, they often possess a remarkable simplifying invariance under changes of scaling length. This statistical self-similarity is the essential quality of fractals in nature. It may be quantified by a fractal dimension which need not be an integer.

The fractal or Hausdorff dimension [1] has recently been used in characterizing disordered materials such as the percolation cluster at the percolation threshold [2] or random particle aggregates [3]. Such characterization has also been used for the electronic wavefunctions in disordered systems such as the pseudo-random Anderson model or tight-binding model [4,5].

In this paper, we explore the fractal nature of the quasi-energy eigenfunctions for an N-level quantum system perturbed by intense polychromatic fields. This provides a new way for characterizing the non-linear dynamical interaction of atoms and molecules with strong quasiperiodic laser fields.

Without loss of generality, let us consider the non-linear response and multiphoton excitation dynamics of a two-level (or spin 1/2) system driven by two intense linearly polarized monochromatic fields. The Schrödinger equation, within the electric dipole approximation, can be written as ($\hbar = 1$)

$$\mathcal{H}(r, t) \psi(r, t) = 0 ,$$  \hspace{1cm} (1)

where

$$\mathcal{H}(r, t) = H(r, t) - i \frac{\partial}{\partial t}$$  \hspace{1cm} (2)

with

$$H(r, t) = H_0(r) + \sum_{i=1}^{N} V_i(r, t)$$  \hspace{1cm} (3)

and

$$V_i(r, t) = -\mu \cdot E_i \cos(\omega_i t + \phi_i) .$$  \hspace{1cm} (4)

$H_0$ is the unperturbed Hamiltonian with orthonormal eigenstates ($\mid \alpha \rangle$, $\mid \beta \rangle$) and eigenvalues ($E_\alpha$, $E_\beta$), $\mu$ is the electric dipole moment operator, and $E_i$, $\omega_i$, and $\phi_i$ are, respectively, the (peak) amplitude, the frequency,
and the initial phase of the $i$th monochromatic field ($i = 1, 2$). The two field frequencies $\omega_1$ and $\omega_2$ are assumed to be incommensurate. For the Hermitian operator $\mathcal{H}(r, t)$, one can introduce the composite Hilbert space $R \otimes T_1 \otimes T_2$. The spatial part $R$ is spanned by $(|\alpha \rangle, |\beta \rangle)$, and, in the two-mode temporal part, $T_1$ is spanned by $\exp(in_1 \omega_1 t)$ and $T_2$ by $\exp(in_2 \omega_2 t)$, where $n_1, n_2 = 0, \pm 1, \pm 2, \ldots$ are the Fourier indices. The time-dependent Schrödinger equation, eq. (1), can be solved exactly by means of the many-mode Floquet theory [6,7].

Thus in the present two-mode case, the time-evolution operator in its matrix form can be written as

$$U_{n_1n_2}(t, t_0) = \langle \gamma_1 | \hat{U}(t, t_0) | \gamma_2 \rangle = \sum_{n_1n_2} \langle \gamma_1 | n_1n_2 | \exp[-i\hat{H}_F(t-t_0)] | \gamma_2 \rangle \exp[i(n_1\omega_1 + n_2\omega_2)t],$$

$$\gamma_1, \gamma_2 \equiv \alpha \text{ or } \beta.$$  (5)

Here $\hat{H}_F$ is the time-independent two-mode Floquet Hamiltonian defined in the generalized Floquet state basis [6],

$$|n_1n_2\rangle \equiv |\gamma\rangle \otimes |n_1\rangle \otimes |n_2\rangle$$

and satisfies the infinite-dimensional eigenvalue equation,

$$\sum_{j,k} \langle \gamma_1 | n_1n_2 | \hat{H}_F | \gamma_2 \rangle k_1k_2 = H[\gamma]_{n_1n_2} + (n_1\omega_1 + n_2\omega_2)\delta_{n_10} \delta_{n_20},$$

where

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with

$$H[\gamma]_{n_1n_2} = E_1 \delta_{n_12} \delta_{n_20} + \sum_{i=1}^{2} V^{(i)}_{n_1n_2} (\delta_{n_21} + \delta_{n_2-1}) \delta_{n_10},$$

$$V^{(i)}_{n_1n_2} = -\frac{1}{2} \langle \gamma_1 | \mu \cdot E | \gamma_2 \rangle \exp(i\phi_i)$$

and

$$\delta_{n_20} = \delta_{n_20}, \text{ if } i = 1, \delta_{n_20} = \delta_{n_20}, \text{ if } i = 2.$$  (9b)

The quasi-energy eigenvalues ($\lambda_{\gamma_1\gamma_2n_1n_2}$) and their corresponding eigenvectors ($|\lambda_{\gamma_1\gamma_2n_1n_2}\rangle$) of $\hat{H}_F$ have the following periodic property forms, namely,

$$\lambda_{\gamma_1\gamma_2n_1n_2} = \lambda_{\gamma_100} + n_1\omega_1 + n_2\omega_2$$  (10)

and

$$\langle \gamma_1, n_1 + q_1, n_2 + q_2 | \lambda_{\gamma_1\gamma_2n_1n_2 + q_1n_1 + q_2n_2} \rangle = \langle \gamma_1 | n_1n_2 | \lambda_{\gamma_1\gamma_2n_1n_2 + q_1n_1 + q_2n_2} \rangle.$$  (11)

The eigenvalue equation, eq. (6), can be alternatively converted into the following two-dimensional (Anderson-model-type) iterative equations:

$$b^{(2)} Z_{n_1n_2-1} + b^{(1)} Z_{n_1+1}n_2 = E_{n_1n_2} Y_{n_1n_2} + b^{(1)} Z_{n_1-1}n_2 + b^{(2)} Z_{n_1n_2-1} = \lambda Y_{n_1n_2},$$

$$b^{(2)} Y_{n_1, n_2-1} + b^{(1)} Y_{n_1n_2} = E_{n_1+1}n_2 Z_{n_1} - 1 + b^{(1)} Y_{n_1, n_2} + b^{(2)} Y_{n_1, n_2-1} = \lambda Z_{n_1, n_2-1}.$$  (13)

Here

$$E_{n_1n_2} = E_0 + n_1\omega_1 + n_2\omega_2,$$

and $Y_{n_1n_2}$ and $Z_{n_1n_2}$ are eigenvector components of the quasi-energy eigenfunctions $|\lambda\rangle$ ($\equiv |\lambda_{\gamma_1\gamma_2n_1n_2}\rangle$), namely,
In the current study, the quasi-energy eigenvalues $\lambda$ and eigenvectors $|\lambda\rangle$ were solved directly by numerical diagonalization of a truncated Floquet matrix $\hat{H}_F$ (defined in eq. (7)) whose dimensionality depends upon the laser field strengths. Up to 1000 by 1000 in matrix dimension is used to achieve numerical convergence for the highest field strength ($b^{(1)} \approx 0.5$) used in this study.

Fig. 1 shows an example of the mountainous quasi-energy eigenfunction in the two-dimensional Fourier space $|n_1\rangle \otimes |n_2\rangle$ for the case of $E_g - E_c = 1.0$, $\omega_1 = 1.0$, $\omega_2 = \frac{1}{3}$, $\phi_1 = \phi_2 = 0$, $b^{(1)} = b^{(2)} = 0.25$ (arbitrary units). To see the subtleties of the wavefunction behavior, a logarithmic plot of the modulus of the same eigenfunction is depicted in fig. 2, exhibiting striking self-similar recurrence phenomena around the diagonal region. Fig. 3 displays the cross sections of the eigenfunctions in fig. 2 as a function of $n_2$ for several different (fixed) values of $n_1$. The self-similarity of the recurrence pattern in the $n_2$ direction is apparent and remarkable particularly because the eigenfunctions can vary as much as fifteen orders of magnitude in amplitude. Similar behavior is observed in the $n_1$ direction when $n_2$ is held fixed.

The self-similar and fragmented character of the quasi-energy eigenfunctions suggest that they may be fractal objects. In disordered systems, the fractal dimension $D_r$ of the probability density in a given eigenstate is defined through \cite{4}

\[ A(L) = \int d^D r_0 \rho(r_0) \int_0^L drr^{D-1} \rho(r+r_0) = cL^{D_r}, \]

where $D$ is the Euclidean dimension, $L$ the scaling length, $\rho$ the probability density of the wavefunction, and $c$ is a constant. This definition allows the averaging over all possible choices of the origin, each weighted by the density (or probability) itself. In our current problem, the density correlation function $A(L)$ can be defined for the quasi-energy eigenfunction $|\lambda_{n_1n_2}\rangle$ in the two-dimensional Fourier $|n_1\rangle \otimes |n_2\rangle$ discrete space as

\[ |X_{n_1,n_2}| \]

Fig. 1. Modulus of the quasi-energy state eigenfunctions components $|X_{n_1,n_2}|$ ($= |\langle n_1,n_2,|\lambda\rangle|)$ versus the Floquet Fourier indices $n_1$ and $n_2$. Physical parameters used are $E_a = 0, E_g = 1.0, \omega_1 = 1.0, \omega_2 = \frac{1}{3}, \phi_1 = \phi_2 = 0, b^{(1)} = b^{(2)} = 0.25$ (arbitrary units). The quasi-energy eigenvalue is $\lambda = 0.0406$. 

89
Fig. 2. Logarithm of $|X_{n_1,n_2}|$ versus the Floquet Fourier indices $n_1$ and $n_2$. Physical parameters are the same as in fig. 1.

Fig. 3. Cross sections of $\ln|X_{n_1,n_2}|$ (in fig. 2) as a function of $n_2$ for several fixed $n_1$ values. (a), (c), (e) are $Y_{n_1,n_2} \equiv \langle \alpha n_1,n_2 | \lambda \rangle$ components, while (b), (d), (f) are $Z_{n_1,n_2} \equiv \langle \beta n_1,n_2 | \lambda \rangle$ components. Parameters are the same as in fig. 1. Notice the striking self-similarity of the wavefunction components.
\[ A(L) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{\gamma = \alpha, \beta} \left| \left< \gamma' n_1 n_2 | \lambda_{y m_1 m_2} \right> \right|^2 \sum_{k_1=-L}^{L} \sum_{k_2=-L}^{L} \sum_{\gamma' = \alpha, \beta} \left| \left< \gamma'' n_1 + k_1, n_2 + k_2 | \lambda_{y m_1 m_2} \right> \right|^2. \]  

Fractal character is obtained when \( A(L) \propto (2L+1)^{D_f} \), where \( L \) is now the integer scaling length in the Fourier space.

Another useful measure of the dimension of fractal objects is relevant to the information content of the system called the entropy dimension \( D_s \) \[8\]. In this case, we can define

\[ S(L) = -\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{\gamma = \alpha, \beta} \left| \left< \gamma' n_1 n_2 | \lambda_{y m_1 m_2} \right> \right|^2 \times \sum_{k_1=-L}^{L} \sum_{k_2=-L}^{L} \sum_{\gamma' = \alpha, \beta} \left| \left< \gamma'' n_1 + k_1, n_2 + k_2 | \lambda_{y m_1 m_2} \right> \right|^2 \ln \left| \left< \gamma'' n_1 + k_1, n_2 + k_2 | \lambda_{y m_1 m_2} \right> \right|^2. \]  

The entropy dimension \( D_s \) is obtained when \( S(L) \propto (2L+1)^{D_s} \). Note that owing to the normalization of the quasi-energy eigenfunction \( |\lambda_{y m_1 m_2}\rangle \),

\[ \sum_{n_1 n_2} \sum_{\gamma} \left| \left< \gamma' n_1 n_2 | \lambda_{y m_1 m_2} \right> \right|^2 = 1, \]  

it can be shown that

\[ \lim_{L \to \infty} A(L) = 1 \quad \text{and} \quad \lim_{L \to \infty} S(L) = \text{constant} \neq 1. \]

In fig. 4, we plot \( \ln A(L) \) versus \( \ln(2L+1) \) for several different fields strengths for quasi-energy states in the reduced zone \[6\]. It is seen \( \ln A(L) \) versus \( \ln(2L+1) \) follows a straight line quite well and then bends over. The point of bending is related to the localization length of the quasi-energy eigenfunction in the \( (n_1, n_2) \) Fourier space. Beyond the localization length, the eigenfunction decays rapidly. As expected, the localization length extends when the field strength increases. The fractal dimension \( D_f \) is obtained from a least-squares fitting of the slope of the linear portion of the graphs in fig. 4. Similar behavior is observed in the plot of \( \ln S(L) \).

![Fig. 4. Plot of ln A(L) versus ln(2L+1) for several different field strengths b for quasi-energies \( \lambda \) in the reduced zone. (a) b=0.05, \( \lambda = -0.0182 \), (b) b=0.1, \( \lambda = -0.0145 \), (c) b=0.2, \( \lambda = 0.0269 \). Other physical parameters are the same as in fig. 1.](image)

![Fig. 5. Fractal (D_f) and entropy (D_s) dimensions of quasi-energy states versus the field strength parameter b. These dimensions were obtained by averaging over ten different quasi-energy states in the reduced zone. Other physical parameters are the same as in fig. 1.](image)
versus \( \ln(2L+1) \) and the entropy dimension \( D_s \) is determined. In general, we found \( D_t < D_s < D \), where \( D \) is the topological dimension, which is two for the \((n_1, n_2)\) two-mode space. From this study, we assert that for any discrete quantum system driven by \( M \)-mode polychromatic fields

\[
V(t) = -\sum_{\gamma=1}^M u\gamma E\gamma \cos(\omega\gamma t + \phi\gamma),
\]

the dimension measure of the quasi-energy eigenfunctions obeys the relationship

\[
D_t < D_s < D.
\]

Also significant to study is the behavior of the fractal dimension when the laser field strengths are varied. In fig. 5, both \( D_t \) and \( D_s \) are presented as a function of the field strength \( b \) (we set \( b^{(1)} = b^{(2)} = b \) here). For each point in \( b \), the dimension is obtained by averaging \( D_t \) or \( D_s \) over ten arbitrary quasi-energy states in the reduced zone where the quasi-energies \( \lambda \approx 0 \). To within 5\%, all the quasi-energy eigenfunctions give rise to the same \( D_t \) or \( D_s \). Fig. 5 shows that both \( D_t \) and \( D_s \) increase as the field strength parameter \( b \) is increased. In the limit of \( b \to \infty \), the quasi-energy eigenfunction delocalizes over the whole \((n_1, n_2)\) Fourier space, and both \( D_t \) and \( D_s \) will approach the topological dimension \( D (= 2) \).

In conclusion, we have shown that the quasi-energy eigenstates of quantum systems perturbed by intense quasiperiodic or polychromatic fields exhibit fractal nature in the temporal Fourier space. We have presented a method for the calculation of the fractal and entropy dimensions. We are currently extending the method to the study of the fractal characteristics in non-integrable atomic and molecular quantum systems whose classical limits are chaotic.

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References